This conception of dominance of linked factors to account for the facts as so far known does not preclude the possibility of a physiological effect resulting from hybridization apart from hereditary factors if such an effect can be demonstrated. It simply coördinates the existing knowledge of heredity to give a comprehensible view of the way in which heterosis may be brought about.

[1] Contribution from the Connecticut Agricultural Experiment Station and the Bussey Institution of Harvard University.

[2] East, E. M., *Connecticut Agric. Exp. Sta. Rep.*, 1907, 1908, (419–428); *Amer. Nat., Lancaster, Pa.*, **43**, 1909, (173–181).

[3] Shull, G. H., *Amer. Breeders Assoc. Rep.* **4**, 1908, (296–301); *Amer. Nat., Lancaster, Pa.*, **45**, 1911, (234–252).

[4] East, E. M., and Hayes, H. K., *U. S. Dept. Agric., Bur. Plant Ind. Bull.* No. 243, 1912.

[5] Keeble, F., and Pellew, C., *J. Genetics, Cambridge*, **1**, 1910, (47–56).

CHEMICALLY INDUCED CROWNGALLS

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In 1911 in Bulletin 213 on "Crown Gall of Plants: its Cause and Remedy" (Bureau of Plant Industry, U. S. Dept. of Agric.) I expressed the conviction that while the disease was clearly due to *Bacterium tumefaciens* we would eventually be able to go a step farther (l.c., p. 175) and determine just what by-products of the organism were the direct cause of the over-growth. With this end in view, on several occasions I prepared flask cultures of the organism for use of the chemist of the Department and with substances said by him to be present in the culture flasks and absent from the controls I have recently made experiments which tend to confirm my earlier supposition and expectation.

It is not maintained for a moment that these are the only substances that are able to cause overgrowths in plants but only that they are the most interesting ones in that they are the products of a cancer parasite, or, if one prefers so to express it, of a schizomycete which is the cause of a plant tumor possessing many features in common with animal cancers.

The substances produced by *Bacterium tumefaciens* in very simple culture media, *i.e.*, in flasks of distilled water containing 1% dextrose and 1% peptone with a little calcium carbonate added to neutralize any acids formed and thus to favor long continued growth since the crown gall organism is very sensitive to its own acid products, are—alde-

hyd, ammonia, amines, alcohol, acetone, acetic acid and formic acid, to which I think I may add traces of CO_2.

These substances, it will be observed, are, for the most part, just those compounds which Jacques Loeb and others have observed to be the most efficient in starting growth in unfertilized eggs of the sea urchin, to wit ammonia, amines and fatty acids (vide Loeb: *Artificial Parthenogenesis and Fertilization.* The University of Chicago Press, 1913). Their action in all probability is purely physical, that is, due to withdrawal of water from neighboring cells by increase of osmotic pressures whereupon the cells so acted upon begin to grow, at least it is possible to obtain the same phenomenon in plants with a great variety of substances, not the product of parasites and not likely to come into contact naturally with the growing tissues.

Thus far I have made no experiments with aldehyd or formic acid and have only just begun to experiment with acetone, but experimenting with the other substances named I have obtained from young tissues a prompt response in the form of over-growths. At first I used water dilutions of these substances painted on or injected, but eventually I used their vapors with marked success. The ethyl alcohol, however, was used mixed with acetic acid, and I do not yet know its unmixed effect. These experiments have been made on several kinds of plants subject to crowngall, especially on Ricinus, cauliflower and Lycopersicum.

The tumors obtained have been small because only a single application was made and consequently the stimulus was soon exhausted, but there is no reason to doubt that a continued application of these substances in high dilution, after the manner of the parasite itself, would result in tumors essentially like those occurring naturally in crowngall, or resulting from our bacterial inoculations.

These tumors are either vascularized hyperplasias, mixed hypertrophy and hyperplasia, or simple hypertrophies. In them the cells are much more closely compacted than the parent cells and free from chlorophyll. The cells of the hypertrophies are frequently a hundred times the volume of the cells from which they have originated. In the acetic alcohol tumors there has been a great increase in the number of the cells, *i.e.*, the development of a true hyperplasia, while in the hypertrophies the component cells appear to be the original cells greatly enlarged.

Curious vascular displacements and duplications have also been ob-

tained including in one instance an entire extra vascular cylinder in the pith of Ricinus.

A fuller account accompanied by photographs and photomicrographs will be published in the *Journal of Agricultural Research.*

[Since this was written I have also obtained small overgrowths on cauliflower leaves with both formaldehyde and formic acid but not with vapor of ethyl alcohol or of acetone. E. F. S.]

DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

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Communicated by E. H. Moore, March 12, 1917

The present note contains a brief summary of a paper with the same title which is about to appear in the *Transactions of the American Mathematical Society.*

The equations of motion of the dynamical system under consideration are taken in the variational form due to Lagrange

$$\frac{d}{dt} L_{x'} - L_x = 0, \quad \frac{d}{dt} L_{y'} - L_y = 0,$$

where x, y represent the two coördinates of the system and where L is a quadratic function of their time derivatives x',y'. By an appropriate change of variables the principal function L is reduced to

$$L = \tfrac{1}{2}(x'^2 + y'^2) + \alpha x' + \beta y' + \gamma,$$

and the equations of motion then take the simple form

$$x'' + \lambda y' = \gamma_x, \qquad y'' - \lambda x' = \gamma_y, \qquad (\lambda = \alpha_y - \beta_x),$$

involving only the two arbitrary functions λ, γ, of x, y. In the *reversible* case, i.e., the case when linear terms are lacking in L, we have $\lambda = 0$. The normal form just written is known in this special case, but is new in the general case so far as I have been able to determine. Any conformal transformation of the variables x, y joined with the corresponding transformation of t leaves this normal form unaltered.

By means of the transformation theory thus obtained it is established that a necessary condition for the existence of an integral linear in x', y' is that the curves λ/γ = const. form an isothermal family in the x, y-plane When this condition is satisfied suppose the particular transformation of the variables x, y to be made which takes the isothermal family into the family y = const.; if and only if a linear integral exists will the resulting functions λ, γ become functions of y alone; and in this case

the equations of motion can be directly integrated. As a second application of the transformation theory the equations of displacement are derived.

For reversible problems it is known that the equations of motion can always be interpreted as those of a particle constrained to move on a fixed smooth surface. In the irreversible case I have been able to show that x, y may be regarded as the coördinates of a particle constrained to move in a smooth surface which rotates uniformly about a fixed axis and carries with it a conservative field of force.

Among all types of orbits the most fundamentally important are those which are periodic.

The existence of periodic orbits is intuitively evident in the reversible case, when the orbits may be looked upon as geodesics. If proper restrictions are imposed a closed geodesic will exist along which the arc length is a minimum, and this geodesic will correspond to a periodic orbit of minimum type. Orbits of minimum type may also be derived by means of an interesting criterion for the reversible case due to Whittaker[1] first rigorously established by Signorini.[2] Notwithstanding the fact that the integral analogous to arc length may change sign in the irreversible case Whittaker stated the direct formal extension of his criterion for this case. I have established that such an extension is legitimate if λ is of one sign, but not otherwise. This restriction on λ is fulfilled in the restricted problem of three bodies.

Unfortunately, as Poincaré pointed out, only unstable periodic orbits can be of minimum type.

Another method may be employed to find a large class of stable periodic orbits which I call of *minimax type.* This entirely new method may be stated in a special case as follows: There is a minimum length of string, constrained to lie in a given surface of genus 0, which may be slipped over that surface. In some intermediate position the string will be taut and will then coincide with a closed geodesic.

Poincaré has proved that a closed geodesic exists on any *convex* surface by an entirely different method.

A third method for the discovery of periodic orbits is that of analytic continuation. Hitherto the application of this method has been limited by the restriction that the variation of the parameter involved be 'sufficiently small.' This restriction turns out to be unnecessary in the reversible case if the orbits near any orbit always intersect it. If the conditions

$$\lambda > O,\ \gamma > 0,\ \Delta\ (\log\gamma)\ > 0$$

hold in the irreversible case the restriction is also unnecessary.

An application of periodic orbits which is fundamental lies in the construction of surfaces of section. The existence of a ring-shaped surface of this sort was noted by Poincaré in the restricted problem of three bodies and allied dynamical problems. The results of my paper show that such surfaces exist in a wide variety of cases and may be of any genus and possess any number of boundaries. By means of a surface of section the dynamical problem reduces to a one-to-one analytic area-preserving transformation of the surface of section into itself.

Periodic orbits correspond to invariant points of this transformation. Consequently the discovery of further periodic orbits hinges upon the proof of the existence of such invariant points. Two theorems concerning such points are proved by me.

The first of these theorems is the following: If a surface of genus p admits of a one-to-one analytic transformation into itself which may be looked upon as obtained by a deformation then the excess of completely unstable invariant points over all other types of invariant points is precisely $2p - 2$.

If the transformation is merely restricted to be continuous the same argument shows that there exists at least one invariant point for $p \neq 1$. For the case $p = 0$ this last result constitutes a well-known theorem due to Brouwer.[3]

The dynamical application of the result for the analytic case is to a fundamental equality obtaining between the various types of periodic orbits.

The second theorem is the following: If the region outside of a circle in the plane is transformed into itself in such a way that points on the circumference are advanced in one sense, so that points distant from the center are regressed by more than a fixed positive angle about the center, and so that areas are preserved, then at least two points are left invariant by the transformation. This theorem is to be regarded as the extension of a theorem stated by Poincaré and proved by me.[4]

These two related theorems are used to prove that infinitely many periodic orbits exist whenever a surface of section is at hand.

[1] Whittaker, *Analytical Dynamics*, pp. 376-384, *Cambridge*, 1904.

[2] Signorini, *Palermo, Rend. Circ. Mat.*, **33**, 1912, (187–193).

[3] Brouwer, *Math. Ann., Leipzig*, **69**, 1910, (176–180).

[4] Birkhoff, *Trans. Amer. Math. Soc., New York*, **14**, 1913, (14–22).